DC State Estimation Model-Based Mixed Integer Semidefinite Programming for Optimal PMU Placement

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***Abstract*— Phasor measurement units (PMUs) can make state estimation more accurate by providing synchronized voltage phasor and current phasor measurements. Optimal PMU placement (OPP) minimizes the number of PMUs required for the system to be completely observable. This paper presents a DC state estimation model using mixed integer semidefinite programming (MISDP) approach for the OPP problem. A comparison between MISDP and mixed integer linear programming (MILP) is conducted. Power flow measurements, injection measurements, limited communication facility, and single PMU failure are studied for each approach. A formulation for MISDP-based PMU placement considering a single PMU failure is proposed. The advantages and disadvantages of each formulation are discussed.**

***Index Terms*— Mixed integer linear programming, mixed integer semidefinite programming, optimal PMU placement**

I. INTRODUCTION

Power system security needs to have a real-time monitor for situation awareness of the operating conditions of the system. In a control center, the state estimator deals with the measurements received from the remote terminal units (RTUs) at the substations and gives the best system state variables (voltage phasor at each bus). Those measurements include bus voltages, branch currents, real and reactive power flows, and power injections. Recently, phasor measurement units (PMUs) with time tags from global positioning system (GPS) can provide synchronized phasor measurements of voltages and currents [1], [2]. PMUs provide better situation awareness due to their much faster sampling rate (30 $\sim$ 120 Hz) [3]. When a PMU is placed at one bus, it can measure the voltage phasor of the bus and current phasors of all lines connected to that bus making the system observable [4]. However, installing PMUs at every bus is expensive. Thus, the problem that needs to be addressed is to find the optimal PMU placement (OPP) that can make the system observable using PMUs at certain buses.

In the literature, optimal PMU placement is solved using two techniques which are heuristic-based and mathematical programming-based. Mathematical programming-based methods are developed with two main formulations: MILP-based and nonlinear programming-based. MILP algorithm to solve the OPP problem is proposed in [2], [4], while nonlinear programming algorithm is presented in [5], [6]. In [7], a comparison between MILP and nonlinear programming is conducted. Another effective mathematical programming-based method to solve the OPP problem is mixed integer semidefinite programming (MISDP) subjected to linear matrix inequality (LMI). This method is based on numerical observability, whereas most of the other methods are based on topological observability which may not ensure numerical observability to execute state estimation successfully [9]. MISDP based on Jacobian matrix of AC state estimation model is proposed in [8], [9].

In this paper, the optimal PMU placement is solved with a constant Jacobian matrix of DC state estimation model using mixed integer semidefinite programming (MISDP) approach. A comparison between this approach and mixed integer linear programming (MILP) is conducted. Several cases including power flow measurements, injection measurements, limited communication facility, and single PMU failure are studied for each formulation. A new MISDP formulation for a single PMU failure is proposed. The advantages and disadvantages are discussed for each method.

The remaining sections are organized as follows. Section II and Section III explain the measurement model and the MILP and MISDP OPP formulations. Section IV investigates the aforementioned four case studies. Section V and Section VI are OPP problem simulation results and conclusion.

II. MEASUREMENT MODEL

Consider the π-model of Bus i and Bus j where a single PMU is installed at Bus i as shown in Fig. 1. The voltage phasor of Bus i is $\widetilde{V}_i = V_i\angle\theta_i$, and the current phasor of Branch $i-j$ is $\widetilde{I}_{ij} = I_{ij}\angle\delta_{ij}$. The rectangular forms of the voltage phasor of Bus i and the current phasor of Branch $i-j$ are $\widetilde{V}_i = V_i\cos\theta_i + jV_i\sin\theta_i = E_i + jF_i$ and $\widetilde{I}_{ij} = I_{ij}\cos\delta_{ij} + jI_{ij}\sin\delta_{ij} = I_{ij,\text{real}} + jI_{ij,\text{imag}}$, respectively. The series admittance of Branch $i-j$ is $y_{ij} = g_{ij} + jb_{ij}$, and the shunt admittance of Bus i is $y_{si} = g_{si} + jb_{si}$. Then the real and reactive power flow measurements from Bus i

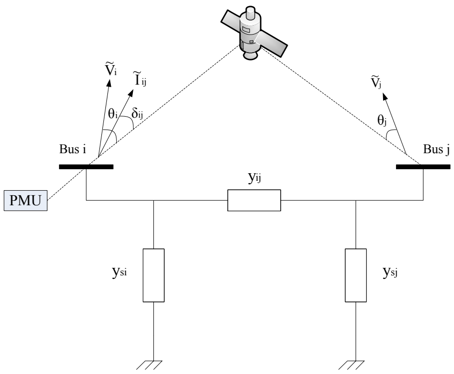

Fig. 1. π-model of Bus i and Bus j

to Bus j and power injection measurements at Bus i can be expressed as follows [10].

$$\begin{aligned} P_{ij}^{\text{meas}} =&(E_i^2+F_i^2)(g_{si}+g_{ij})+E_i(-g_{ij}E_j+b_{ij}F_j) \\ &-F_i(b_{ij}E_j+g_{ij}F_j)+e_{P_{ij}^{\text{meas}}} \end{aligned} \tag{1}$$

$$\begin{aligned} Q_{ij}^{\text{meas}} =&-(E_i^2+F_i^2)(b_{si}+b_{ij})+E_i(b_{ij}E_j+g_{ij}F_j) \\ &+F_i(-g_{ij}E_j+b_{ij}F_j)+e_{Q_{ij}^{\text{meas}}} \end{aligned} \tag{2}$$

$$\begin{aligned} P_i^{\text{meas}} =&(E_i^2+F_i^2)\sum_{j\in \boldsymbol{S}ad_i}(g_{si}+g_{ij})+E_i\sum_{j\in \boldsymbol{S}ad_i}(-g_{ij}E_j \\ &+b_{ij}F_j)-F_i\sum_{j\in \boldsymbol{S}ad_i}(b_{ij}E_j+g_{ij}F_j)+e_{P_i^{\text{meas}}} \end{aligned} \tag{3}$$

$$\begin{aligned} Q_i^{\text{meas}} =&-(E_i^2+F_i^2)\sum_{j\in \boldsymbol{S}ad_i}(b_{si}+b_{ij})+E_i\sum_{j\in \boldsymbol{S}ad_i}(b_{ij}E_j \\ &+g_{ij}F_j)+F_i\sum_{j\in \boldsymbol{S}ad_i}(-g_{ij}E_j+b_{ij}F_j)+e_{Q_i^{\text{meas}}} \end{aligned} \tag{4}$$

where $\boldsymbol{S}ad_i$ is the set of buses adjacent to Bus i.

The real and imaginary parts of current phasor measurement (measured by PMU) from Bus i to Bus j will be as follows [10], [11].

$$I_{ij,\text{real}}^{\text{meas}} = (g_{si}+g_{ij})E_i-(b_{si}+b_{ij})F_i-(g_{ij}E_j-b_{ij}F_j)+e_{I_{ij,\text{real}}^{\text{meas}}} \tag{5}$$

$$I_{ij,\text{imag}}^{\text{meas}} = (b_{si}+b_{ij})E_i+(g_{si}+g_{ij})F_i-(b_{ij}E_j+g_{ij}F_j)+e_{I_{ij,\text{imag}}^{\text{meas}}} \tag{6}$$

The real and imaginary parts of voltage phasor measurement (measured by PMU) at Bus i will be as follows [10], [11].

$$E_i^{\text{meas}} = E_i + e_{E_i^{\text{meas}}} \tag{7}$$

$$F_i^{\text{meas}} = F_i + e_{F_i^{\text{meas}}} \tag{8}$$

The DC measurement model is assumed where $\widetilde{V}_i = 1\angle 0^\circ$ p.u. is the voltage phasor. The branch susceptances will be obtained from MATLAB-based software package MATPOWER, and all shunt elements and branch conductances will be neglected.

Then consider the measurement model of a DC power system state estimation as the following:

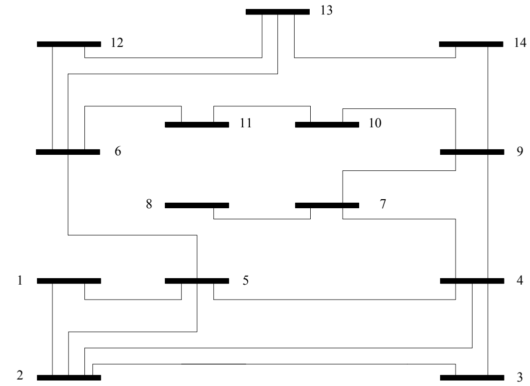

Fig. 2. IEEE 14-bus system

$$z = Hx + e \tag{9}$$

where z is the measurement vector, x is the state vector, H is the constant Jacobian matrix, and e is the error vector.

The linear weighted least squares (WLS) state estimation will be as the following [10]:

$$G\hat{x} = H^T R^{-1} z \tag{10}$$

where $G = H^T R^{-1} H$ is the gain matrix, $\hat{x}$ is the estimated x, and R is the diagonal covariance matrix.

III. Optimal PMU Placement Formulations

A. Mixed Integer Linear Programming

The power system state estimation investigated in this paper is a DC power flow based linear state estimation. The states are the voltage phase angle (θ_i for Bus i) of every bus in the power grid. The measurements are assumed to come from PMUs, which include phase angle of Bus i and the power flow measurements from Bus i to adjacent buses ($j \in ad_i$, where ad_i is the buses adjacent to Bus i). Therefore, if θ_i is measured by the PMU, then θ_j can also be found since P_{ij} is measurable ($P_{ij} = \frac{1}{X_{ij}}(\theta_i - \theta_j)$ where X_{ij} is the reactance of the line). Thus, with a PMU installed at Bus i, Bus i and all its adjacent buses are observable.

Bus i is observable as long as there is at least one PMU installed at Bus i or at its adjacent buses. This requirement can be expressed by an inequality constraint:

$$f_i(x) = x_i + \sum_{j\in ad_i} x_j \geq 1 \tag{11}$$

where x_j is a binary variable to indicate if there is a PMU at Bus j ($x_j = 1$) or not ($x_j = 0$), and $f_i(x)$ is introduced as the observability function for Bus i.

For the IEEE 14-bus system as shown in Fig. 2, the OPP formulation can be expressed as follows.

$$\begin{aligned} \min_x \quad & \sum_{k=1}^{14} x_k \\ \text{subject to:} \quad & f_i(x) \geq 1 \\ & x_i \in \{0,1\}, i = 1,2,\cdots 14. \end{aligned}$$

where

$$
f_i(x) = \begin{cases} f_1 = x_1 + x_2 + x_5 \geq 1 \\ f_2 = x_1 + x_2 + x_3 + x_4 + x_5 \geq 1 \\ f_3 = x_2 + x_3 + x_4 \geq 1 \\ f_4 = x_2 + x_3 + x_4 + x_5 + x_7 + x_9 \geq 1 \\ f_5 = x_1 + x_2 + x_4 + x_5 + x_6 \geq 1 \\ f_6 = x_5 + x_6 + x_{11} + x_{12} + x_{13} \geq 1 \\ f_7 = x_4 + x_7 + x_8 + x_9 \geq 1 \\ f_8 = x_7 + x_8 \geq 1 \\ f_9 = x_4 + x_7 + x_9 + x_{10} + x_{14} \geq 1 \\ f_{10} = x_9 + x_{10} + x_{11} \geq 1 \\ f_{11} = x_6 + x_{10} + x_{11} \geq 1 \\ f_{12} = x_6 + x_{12} + x_{13} \geq 1 \\ f_{13} = x_6 + x_{12} + x_{13} + x_{14} \geq 1 \\ f_{14} = x_9 + x_{13} + x_{14} \geq 1 \end{cases}
$$

The problem is solved using the mixed integer linear programming by MATLAB and *intlinprog* function, and the optimal PMU placement will be on buses 2, 6, 7, and 9.

The generalized mixed integer linear programming formulation is expressed as follows [4].

$$
\min_x \quad \sum_{k=1}^{N} w_k \ x_k \tag{12a}
$$

$$
\text{subject to:} \quad Ax \geq B \tag{12b}
$$

$$
x_i \in \{0, 1\}, i = 1, \cdots, N \tag{12c}
$$

where x_i notates if Bus i has a PMU installed ($x_i = 1$) or not ($x_i = 0$), and the objective function is the minimization of the number of PMUs to be installed. The w_k is the PMU installation cost. Note that all PMUs have the same installation cost ($w_i = 1$) which makes the minimization of the PMU installation cost is equivalent to the minimization of the number of PMUs. The A matrix's element and B matrix are defined as follows.

$$
a(i,j) = \begin{cases} 1, \text{ if Bus } i \text{ and Bus } j \text{ are connected} \\ 1, i = j \\ 0. \text{ if Bus } i \text{ and Bus } j \text{ are not connected} \end{cases}
$$

$$
B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T
$$

B. Mixed Integer Semidefinite Programming

In addition to the mixed integer linear programming, optimal PMU placement can be formulated as mixed integer semidefinite programming (MISDP) subjected to linear matrix inequality (LMI) [8]. MISDP is based on numerical observability, while most of OPP methods are based on topological observability which may not ensure numerical observability in some cases to execute state estimation successfully [9]. Numerical observability in the power system can be achieved when the Jacobian matrix has a full rank, whereas the topological observability can be obtained when the formed spanning tree has a full rank [9], [10]. Numerical observability (MISDP-based) guarantees topological observability (MILP-based) [9], and they are equivalent in case of the linearized DC measurement model where bus voltages and line reactances are assumed to be 1 p.u. since topological observability is not based on branch parameters [10].

Mixed integer semidefinite programming considering the OPP problem can be formulated as follows [9].

$$
\min_x \quad \sum_{i=1}^{N} w_i \ x_i \tag{13a}
$$

$$
\text{subject to:} \quad G(x) = G_0 + \sum_{i=1}^{N} x_i G_i \succ 0 \tag{13b}
$$

$$
x_i \in \{0, 1\}, i = 1, \cdots, N \tag{13c}
$$

where $G(x)$ is a full rank matrix, $G(x) \succ 0$ is the positive definite constraint that ensures the required complete numerical observability, $G_0 = H_0^T R_0^{-1} H_0$ is the gain matrix for the exiting conventional and injection measurements in the power system, and $G_i = H_i^T R_i^{-1} H_i$ is the gain matrix for the voltage phasor and current phasors obtained by the PMU at Bus i. This formulation can be employed to solve DC and AC state estimation models. In this paper, the DC state estimation model is assumed.

To explain the above formulation, the mixed integer semidefinite programming problem is solved for IEEE 14-bus system (Fig. 2) assuming DC state estimation model, and it is compared with the solution of the mixed integer linear programming. Based on IEEE 14-bus system, the mixed integer semidefinite programming will be as the following:

$$
\min_x \quad \sum_{i=1}^{14} w_i \ x_i
$$

$$
\text{subject to:} \quad G(x) = G_0 + \sum_{i=1}^{14} x_i G_i \succ 0
$$

$$
x_i \in \{0, 1\}, i = 1, 2, \cdots, 14
$$

As mentioned above, the DC measurement model is used where the voltage phasor is assumed to be 1 p.u., and all branch susceptances are provided from MATLAB-based software package MATPOWER. All shunt parameters and branch conductances are neglected, and unity power factor is assumed ($Q_{ij}^{\text{meas}} = 0$). Thus, the power flow measurement is $P_{ij}^{\text{meas}} = I_{ij}^{\text{meas}}$. Then the Jacobian matrix H_0 is zero since it is assumed that there is no flow or injection measurements, and H_i can be found as follows.

For Bus 1: $H_1 =$

$$
\begin{array}{c} \\ F_1^{\text{meas}} \\ I_{1-2}^{\text{meas}} \\ I_{1-5}^{\text{meas}} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16.9 & -16.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.5 & 0 & 0 & 0 & -4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

For Bus 2: $H_2 =$

$$
\begin{array}{c} \\ F_2^{\text{meas}} \\ I_{2-1}^{\text{meas}} \\ I_{2-3}^{\text{meas}} \\ I_{2-4}^{\text{meas}} \\ I_{2-5}^{\text{meas}} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \end{array} \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -16.9 & 16.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.1 & -5.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.7 & 0 & -5.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.8 & 0 & 0 & -5.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

The gain matrices of buses 1 and 2 are $G_1 = H_1^T R_1^{-1} H_1$ and $G_2 = H_2^T R_2^{-1} H_2$, respectively. Then the Jacobian and gain matrices are obtained for the rest of the system buses. Note that $F_i = F_1, F_2, \cdots, F_{14}$ represents the system state which is the voltage phasor's angle in this case. Therefore, Bus i's voltage phase angle (system state) and the current flowing on branches connected to that bus can be measured with a PMU installed at Bus i. This problem is solved using MATLAB toolbox YALMIP [12] with an outer approximation solver SCIP [13]. The optimal PMU placement is found to be on buses 2, 6, 7, and 9. Thus, mixed integer semidefinite programming is an effective way to solve the optimal PMU placement problem.

IV. Optimal PMU Placement Case Studies

A. OPP Formulation with Power Flow Measurements

Let's assume that there is a power flow meter on Line ij in the system. When either Bus i's or Bus j's state (phase angle) is known, other bus's state can be found should the power flow measurements from Bus i to Bus j are given.

For the MILP formulation, the observability constraints should be changed considering power flow measurements. In the absence of the power measurements on lines i–j, the observability constraints of the two buses will be given by:

$$f_i = \sum_{k=1}^{N} A_{ik} x_k \geq 1 \tag{15a}$$

$$f_j = \sum_{k=1}^{N} A_{jk} x_k \geq 1 \tag{15b}$$

In the presence of the power measurements on Line k with Bus i and Bus j as the terminal buses, the above two constraints will be merged into the following joint constraint [4]:

$$f_{flow,k} = f_i + f_j \geq 1 \tag{16}$$

The above constraint means that as long as Bus i or Bus j is observable, then the other bus is also observable due to the power flow meter.

Let's assume that there are power flow measurements on lines 2-3, 3-4, 6-11, 6-12, and 7-8 in the IEEE 14-bus system (Fig. 2). Then the joint constraint for power flow measurements on lines 2-3 and 3-4 is obtained from (16) as follows.

$$\begin{aligned} f_{flow,2-3,3-4} &= f_2 + f_3 + f_4 \geq 1 \\ &= x_1 + 3x_2 + 3x_3 + 3x_4 + 2x_5 + x_7 + x_9 \geq 1 \end{aligned}$$

The joint constraint indicates that whenever one of the buses (2,3,4) is observable, the rest are observable due to the meters. Thus, we merged the constraints f_2, f_3, and f_4 into one joint constraint $f_{flow,2-3,3-4}$ to assure a placement of one PMU for one of those buses or their adjacent buses. Then the process is repeated to obtain the joint constraints for power flow measurements on lines 6-11, 6-12, and 7-8.

For the MISDP formulation, the Jacobian matrix H_0 will consider the power flow measurements. Suppose that there are power flow measurements on lines 2-3, 3-4, 6-11, 6-12, and 7-8 in the IEEE 14-bus system (Fig. 2). Then the Jacobian matrix H_0 will be as follows.

$$H_0 =$$

$$\begin{array}{cc} & \begin{array}{cccccccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \end{array} \\ \begin{array}{c} P^{\mathrm{meas}}_{2-3} \\ P^{\mathrm{meas}}_{3-4} \\ P^{\mathrm{meas}}_{6-11} \\ P^{\mathrm{meas}}_{6-12} \\ P^{\mathrm{meas}}_{7-8} \end{array} & \begin{bmatrix} 0 & 5.1 & -5.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.8 & -5.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.9 & 0 & 0 & 0 & 0 & 0 & -3.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5.7 & -5.7 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

From the above formulations, we can see that the MISDP formulation is simple and easy to formulate in the case of power flow measurements compared to the MILP formulation. Table I shows that both formulations provide the optimal PMU placement required for complete observability.

B. OPP Formulation with Injection Measurements

Suppose that Bus ℓ is connected to buses i, j, and k as shown in Fig. 3. Then suppose that there is an injection measurement for Bus ℓ. The power injection of the DC measurement model is related to the phase angles of all buses, θ_i, θ_j, θ_k, and θ_l, as follows.

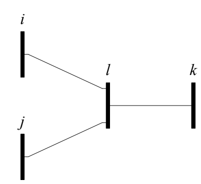

Fig. 3. Four-bus system

$$P_{inj,l} = \frac{\theta_l - \theta_i}{X_{li}} + \frac{\theta_l - \theta_j}{X_{lj}} + \frac{\theta_l - \theta_k}{X_{lk}} = 0 \tag{17}$$

With this measurement and with three of the angles given, the last angle can be found. In other words, the four buses are all observable should three of them are observable due to PMU placement and an injection measurement at Bus ℓ. Note that there is no difference between the zero and nonzero injection measurements [2].

This statement has been described in the MILP formulation in [4]:

$$f_{inj,l} = f_i + f_j + f_k + f_l \geq 3 \tag{18}$$

If only one of the observability constraints (f_i, f_j, f_k, or f_l) is zero, the joint constraint (18) is valid. However, there are two limitations of the joint constraint (18). One of the limitations is that a redundant observability of some buses can result in by adding the observability constraints which may let the joint constraint (18) satisfied with two zero observability constraints [14]. Another limitation is that this joint constraint may not provide the optimal solution in the case of two or more injection measurements with mutual buses [14]. In [14], a modification has been applied to (18)

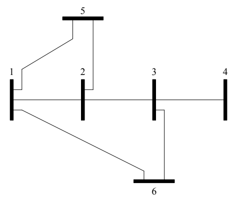

Fig. 4. Six-bus system

to solve these two limitations which results in too many constraints especially if it is solved for a large system. A six-bus system as shown in Fig. 4 is used to simply illustrate the injection measurement problem. Suppose that buses 1 and 3 have an injection measurements. The MILP constraints which are developed in [14] will be as follows.

$$f_{inj,1\&3} = \begin{cases} f_1+f_2+f_3 \geq 1, \quad f_1+f_2+f_4 \geq 1, \quad f_1+f_2+f_6 \geq 1, \\ f_1+f_3+f_6 \geq 1, \quad f_1+f_4+f_6 \geq 1, \quad f_2+f_3+f_5 \geq 1, \\ f_2+f_4+f_5 \geq 1, \quad f_2+f_5+f_6 \geq 1, \quad f_3+f_5+f_6 \geq 1, \\ f_4+f_5+f_6 \geq 1, \quad f_2+f_3+f_6 \geq 1, \quad f_2+f_4+f_6 \geq 1, \\ f_1+f_2+f_6 \geq 1, \; f_1+f_5 \geq 1, \; f_3+f_4 \geq 1, \; f_2+f_6 \geq 1. \end{cases} \tag{19}$$

In contrast, the mixed integer semidefinite programming can solve the problem with a compact and simple formulation. The Jacobian matrix H_0 will consider the injection measurements. Let's assume that buses 1 and 3 have an injection measurements in the six-bus system (Fig. 4) where all branch susceptances are assumed to be -1 per unit. Then the Jacobian matrix H_0 will be as follows.

$$H_0 = \begin{array}{c} \\ P_1^{\text{meas}} \\ P_3^{\text{meas}} \end{array} \begin{array}{c} \begin{array}{cccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \end{array} \\ \begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \end{bmatrix} \end{array}$$

Both MILP and MISDP formulations can provide the optimal PMU placement by installing only one PMU at Bus 2. With the help of injection measurements at buses 1 and 3, buses 4 and 6 can be observable. Note that MILP joint constraint (18) cannot provide this solution; instead, the constraint (19) is used to improve the redundant observability and optimality limitations. Thus, MISDP formulation can solve the injection measurement problem with compact and simple formulation compared to the MILP formulation.

C. OPP Formulation with Limited Communication Facility

The limited communication facility in the substation can prevent the PMU installation due to the lack of data links required to enable the communication between PMUs and the control center. This problem can affect the installation cost of the PMU to be much higher [15]. Thus, a high installation cost w_i will be assigned to the bus that has a limited communication facility for both MILP and MISDP. Consequently, the high installation cost will exclude the limited communication facility buses from the optimal set [7]. Let's assume that there are limited communication facilities at buses 2 and 9 on the IEEE 14-bus system (Fig. 2). Then high installation costs ($w_i = 10^9$) are assigned to buses 2 and 9, whereas installation costs of the other buses are kept as $w_i = 1$.

D. OPP Formulation with Single PMU Failure

Although PMUs are reliable devices, failure of a single PMU is possible. Therefore, to protect the system from losing one PMU and leaving the system unobservable, the optimal PMU set is divided into two sets which are main set and backup set. The main set is the set obtained without a PMU failure, while the backup set is the set that we are going to obtain. For MILP formulation, every bus in the system is going to be observed by two PMUs which modify the right hand side of the inequality constraints to be two instead of one [16]. Also, the backup set of the MILP formulation can be obtained by removing x_i and x_j terms that represent the main set. The main set of the IEEE 14-bus system (Fig. 2) is achieved by solving the problem without considering a PMU failure as in Section III-A. Then the main set for the MILP is the following: $\{2, 6, 7, 9\}$. Therefore, all of the terms x_2, x_6, x_7, and x_9 are removed from the MILP constraints to obtain the backup set.

On the other hand, the main set of the MISDP is provided by solving the problem without considering a PMU failure as in Section III-B. Then the backup set for the MISDP formulation is going to be obtained as follows.

$$\min_{x} \quad \sum_{i=1}^{N} w_i \; x_i \tag{20a}$$

$$\text{subject to:} \quad G(x) = G_0 + \sum_{i=1}^{N} x_i G_i \succ 0 \tag{20b}$$

$$x_i = 0, \forall i \in \mathbf{M} \tag{20c}$$

$$x_i \in \{0, 1\}, i = 1, \cdots, N \tag{20d}$$

where $\mathbf{M}$ is the main set, and constraint (20c) means that the PMUs are already installed to the main set buses. Therefore, the backup set will be generated to solve the single PMU failure problem.

This will ensure that the same bus will not pick up more than one time. Thus, the backup set will keep the system observable when a single PMU failed.

V. OPP Problem Simulation Results

DC state estimation model using MISDP is presented for the OPP problem, and a comparison between MILP and MISDP is conducted. Power flow measurements, injection measurements, limited communication facility, and single PMU failure are formulated for the two approaches. A new formulation for a single PMU failure using MISDP is presented. The MILP optimization problem is solved by MATLAB *intlinprog* function, and the MISDP optimization problem is solved using MATLAB toolbox YALMIP [12] with an outer approximation solver SCIP [13]. DC model is used for the MISDP formulation with standard deviations of 0.0076, 0.016, and 0.0001 for power flows, injections, and phasor measurements (voltage and current), respectively. A standard deviation of 0.00002 is taken for zero injections. Table I presents the comparison between MISDP and MILP. From this table, we can see that case 1 compares between the two formulations without conventional and injection measurements, while cases 2-5 consider power flow and/or injection measurements. The number of PMUs in both formulations is reduced in case of power flow or injection measurements due to the measurement meters. Table II shows the comparison between the two formulations in case of limited communication facility and single PMU failure. Note that the number of PMUs is increased in the limited communication facility and single PMU failure contingency cases. Another optimal set is provided as a backup set which would be very expensive to install. Also, it should be noted that the MISDP can provide the minimum number of PMUs as the MILP but at different locations in some cases.

Both MILP and MISDP are effective ways to solve the OPP problem, and each formulation has its advantages and disadvantages. The MILP formulation has less computational time compared to MISDP formulation as shown in Fig. 5. In the meantime, MISDP CPU time is considered reasonable since the Jacobian matrix is used. Also, MISDP has less CPU time when conventional and/or injection measurements are taken into account. MISDP is more compact and simple for problem formulation related to power flow and injection measurements as explained in Section IV. Further, MISDP can solve DC or AC state estimation models.

TABLE I. COMPARISON RESULTS OF MILP AND MISDP FOR OPP

Case	Optimal Set	Power Flows	Injections	PMU Placement Bus	
				MILP	MISDP
1	4	—	—	2,6,7,9	2,6,7,9
2	3	2-3,3-4,6-11,6-12,7-8	—	2,9,12	5,9,14*
3	3	—	7	2,6,9	2,6,9
4	3	—	8,11,13	2,4,6	1,4,6*
5	2	2-3,3-4,6-11,6-12,7-8	8,11,13	5,9	5,9

* Although the two formulations give different results, both guarantee observability.

TABLE II. COMPARISON RESULTS FOR OPP CONTINGENCIES

Case	Contingency & Location	Power Flows	Injections	PMU Placement Bus	
				MILP	MISDP
6	Limited Communication at 2,9	—	—	1,3,7,10,13	4,5,7,10,13
7		2-3,3-4,6-11,6-12,7-8	8,11,13	5,8,14	5,8,14
8	Single PMU Failure at 2,6,7,9	—	—	2,6,7,9 (Main) + 4,5,8,11,13 (Backup)	2,6,7,9 (Main) + 1,4,8,10,13 (Backup)
9	Single PMU Failure at 5,9	2-3,3-4,6-11,6-12,7-8	8,11,13	5,9 (Main) + 2,7,12 (Backup)	5,9 (Main) + 2,4,6 (Backup)

VI. CONCLUSION

DC state estimation model-based MISDP formulation for the OPP problem is presented for complete observability. This approach is compared with the MILP formulation. The MISDP is based on numerical observability which depends on branch parameters and system state, and it has a compact and simple formulation in case of power flow and injection measurements compared to the MILP formulation. Several observability contingencies, which are power flow measurements, injection measurements, limited communication facility, and single PMU failure, are discussed for both approaches. The advantages and disadvantages of each formulation are presented.

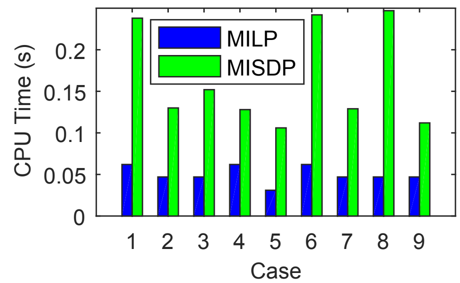

Fig. 5. MILP and MISDP CPU Time Comparison

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