Abstract—With the rise of various distributed energy resources into the distribution networks, an electricity market, similar to that of the transmission level, is foreseen to evolve in the near future, aiming to indulge and financially encourage the private resources to participate in the balance and resiliency of the system. This paper adopts the locational marginal pricing, a well-known scheme to quantify nodal pricing, and analyzes the signals sent to future-market participants to meet the demand, reduce losses, and alleviate congestion. The DLMPs derived in this paper are based on an AC OPF, convexified using semidefinite programming (SDP) relaxation, which accounts for structural characteristics of the distribution networks. Thorough case studies with a multi-period optimization are performed on the IEEE 37-bus feeder with the inclusion of distributed generators and a static-Var compensator. The unbalanced nature of distribution loads and lines induces disparate DLMPs at each phase. The analysis also contrasts passive and active distribution networks giving more emphasis to the contribution and incentives of the latter. The active networks show increased penetrations of end-users’ renewable supply are interpreted as the shadow prices that reflect on the reduction of the entire consumer payment.

Index Terms—distribution locational marginal pricing, multiphase systems, semidefinite programming

I. INTRODUCTION

The modern distribution network is undergoing an unprecedented reformation, thanks to the increased deployment of distributed energy resources (DERs) in the form of distributed generators (DGs), distributed storage (DS), microgrids (MGs), aggregators managing fleets of electric vehicles (EVs), or groups of prosumers, i.e. residential rooftop photovoltaic panels (PVs). Potential benefits offered by DERs are vast. For some, they contribute to load profile betterment (peak shaving/valley filling), support nodal voltages, alleviate line congestions, and defer utility expansions. However, high DER penetrations with ill-managed operation could fail to deliver the desired outcome, causing sharp voltage fluctuations and supply-demand imbalance. That being said, a replication of the transmission-level locational marginal pricing (LMP) mechanism is; therefore, necessary. The locational marginal price is the marginal cost of supplying the subsequent unit of demand at a bus. The price signals differ spatially and temporally, and are used to incentivize DER to balance supply-demand, support voltage, and minimize system losses.

The aforesaid process entails the establishment of a third-party operator -distribution system operator (DSO)- to collect bids from participants including ISO at the wholesale level, control volt-var and DER assets, and settle the market, while abiding by the system physical and security constraints. Market is cleared in real time or with a day-ahead point forecast of loading and substation/DER supply.

The topic has garnered many researchers’ attention. [1] developed DLMPs in the presence of DGs. DLMP was developed in [2], [3] to effectively alleviate congestions caused by high penetrations of EV loads. In [4], the branch flow model (BFM), convexified using second order conic programming (SOCP), was used to calculate DLMPs accounting for voltages, losses and active/reactive powers. In [5], a linearized BFM model aided with loss factors was used to unbundle DLMPs in a multi-period optimization, after running an MISOCP-based BFM to obtain solutions of discrete variables, i.e. switched capacitor banks. Most of the existing literature therein either i) exercises a DC approximation of the OPF model, which ignores system losses or, in the best case, overlooks voltages and reactive powers, essential components to power quality and system efficient operation [2], [3], or ii) abstracts the fact that distribution networks have unbalanced loads and line impedances with phase-to-phase coupling [1], [4], [5]. Hence, the per-phase representation does not necessarily exhibit the system structure. [6] computes active-power DLMPs using a three-phase OPF that assumes a nonconvex model. In our project, we adopt the SDP-relaxed BFM for multiphase systems proposed by Gan and Low [7]. Compared to other SDP relaxation models [8], the model explores the sparsity characteristic of the network, and has a lower-dimension matrix variables. The benefit of the SDP relaxation is a guaranteed global optimum. The DLMPs found are convex-hull prices.

Our contribution is a multi-period optimization of a multiphase system that calculates the active- and reactive-power DLMPs at each bus phase, obtained from the Lagrange multiplier associated with the power balance equations.

II. FORMULATION

In this section, we review the BFM formulation presented in [7], along with system limits and DER/Var device models.
Fig. 1: Variables in SDP relaxation of three-phase OPF.

A. Notation

A radial distribution system can be modeled as a tree-like graph comprised of nodes and directed branch segments, \( G(N, E) \), where \( N \) and \( E \) correspond to the sets of buses and lines; respectively. The root (substation bus) is denoted as 0 with known voltage magnitudes, \( N = \{0, 1, \ldots, n\} \). Then, \( N^+ = N - \{0\} \) is the set of descent buses from substation. Also, lines are indexed by buses to which they are directed, thus \( E = N^+ \times \{1, \ldots, n\} \). Therefore, only \( N^+ \) is used for brevity. Further, we index the distinct parent of bus \( i \) as \( \rho(i) \), and the set of its children buses as \( \delta(i) \).

A phase set of bus \( i \) is denoted as \( \Phi_i \), thus \( \Phi \subseteq \{a, b, c\} \). Hence, if three-phase bus \( i \) is a parent of two-phase bus \( j \), then \( \Phi_j \subset \Phi_i \). A phase element index of a vector variable is subscripted with \( \phi \), whereas those of a matrix diagonal are subscripted with \( \Phi \).

B. Branch Flow Model (BFM)

Equations for branch \( \rho(i) \rightarrow i \), assuming \( \Phi_i = \Phi_{\rho(i)} \), are as follows.

1) Ohm’s Law:

\[ V_i = V_{\rho(i)} - z_i I_i \]

(1)

where \( V_i, V_{\rho(i)}, \) and \( I_i \in \mathbb{C}^{\Phi_i} \), while \( z_i \in \mathbb{C}^{\Phi_i \times \Phi_i} \). By multiplying both sides by their Hermitian transposes, and defining \( v_i = V_i V_i^H, v_{\rho(i)} = V_{\rho(i)} V_{\rho(i)}^H, S_i = V_{\rho(i)} I_i^H \) and \( \ell_i = I_i I_i^H \), then (1) can be rewritten as

\[ v_i = v_{\rho(i)} - (S_i z_i^H + z_i S_i^H) + z_i \ell_i z_i^H \quad \forall \ i \in N^+ \]

(2)

where \( v_i, v_{\rho(i)}, \ell_i, \) and \( S_i \in \mathbb{C}^{\Phi_i \times \Phi_i} \) with diagonal entrees of \( v_i, v_{\rho(i)} \) and \( \ell_i \) as squared magnitudes, and off-diagonal as mutual complex elements. The two-node system in Fig. 1 depicts the relationship, where \( |\Phi_i| = |\Phi_{\rho(i)}| = 3 \).

2) Power Balance: For each \( \rho(i) \rightarrow i \rightarrow k \) where \( k \in \delta(i) \), to interpret the power balance at bus \( i \), (1) is multiplied by \( I_i^H \)

\[ V_i I_i^H = V_{\rho(i)} I_i^H - z_i I_i I_i^H \]

(3)

\[ V_i \left( \sum_{k \in \delta(i)} I_k^H - I_i^H \right) = S_i - z_i \ell_i \]

(4)

where \( I_i \) is the current injection at bus \( i \). As a result, the power balance at bus \( i \) is the diagonal of (4), which can be expressed as

\[ \sum_{k \in \delta(i)} \text{diag}(S_k) - s_i = \text{diag}(S_i - z_i \ell_i) \quad \forall \ i \in N^+ \]

(5)

\( s_i \in \mathbb{C}^{\Phi_i} \) is the net power injection, which is the subtraction of load from the power injected/absorbed via DERs and Var devices. PV/wind distributed generators, static-Var compensators and inelastic loads are considered in this paper. Thus

\[ s_i = s_i^g + s_i^{svc} - s_i^L \]

(6)

For \(|\Phi_i| < 3\), entrees of diag(\( \ell_i \)) diag(\( S_i \)), and impedances that correspond to the missing phase, \( \phi_{\text{missing}} \), can simply be forced to zeros.

3) PSD and Rank-1 Matrix: To close the gap and set the relationship between actual electrical components and surrogate variables, the following positive and rank one matrix is defined

\[ X_i = \begin{bmatrix} v_{\rho(i)} & V_{\rho(i)}^H \end{bmatrix} = \begin{bmatrix} v_{\rho(i)} & S_i^H \end{bmatrix} \begin{bmatrix} I_i \end{bmatrix} \]

\[ X_i \geq 0 \quad \forall \ i \in N^+ \]

(7)

\[ \text{rank}(X_i) = 1 \quad \forall \ i \in N^+ \]

(8)

4) Distributed Generators: It is essential that inverters of both PVs and wind be equipped with curtailment capability in order for the DSO to dispatch an appropriate amount of generated active power for market clearing. DGs are also designed to operate in the first and fourth quadrants to supply/absorb reactive power with a minimum allowable power factor (PF) for Var compensation and voltage support. The set \( DG \) generalizing the PV and wind turbine power sets, \( DG = PV + WT \), and the inequality limit constraint are as follows

\[ DG = \{s_i^g \in \mathbb{C}^{\Phi_i} \mid 0 \leq \text{real}(s_i^g) \leq P_i^g - \text{real}(s_i^g) \leq \text{imag}(s_i^g) \leq \text{imag}(s_i^g)\} \quad \forall \ i \in N_g \]

(9)

\[ a = \sqrt{1 - \text{PF}^2} / \text{PF} \]

(10)

where \( P_i^g \) is the maximum power generation of solar or wind, and \( a \) forces reactive power to a fixed leading/lagging PF [5]. Additionally, \( \omega \in \mathbb{R}^{\Phi_i} \) is defined to represent the penetration level of the DG (measured by their peak with respect to the rated load). For a number of PVs installed, as an example, the per-phase peak power from a single PV is

\[ P_{i}^{\Phi \omega} = \omega \left( \sum_{i \in N^+} \sum_{\phi \in \Phi_i} \text{real}(s_i^\Phi) / |PV| \times |\Phi_i| \right) \quad \forall \ i \in PV \]

(11)

Elements of the resultant \( P_{i}^{\Phi \omega} \) are then subjected to temporal variations of power availability. A similar approach is used for wind turbines.

5) Static-Var Compensators: The DSO can improve the voltage profile by dispatching SVCs to either generate or absorb reactive powers. The SVC’s variable limit is a continuous inequality constraint

\[ SVC = \{s_i^{svc} \in \mathbb{C}^{\Phi_i} \mid \text{real}(s_i^{svc}) = 0 \}

\[ -q \leq \text{imag}(s_i^{svc}) \leq q \] \quad \forall \ i \in N_{svc} \]

(12)
6) Voltage Limits: Except for the substation node \((v_0 = V_0V_0^H)\), ±5% of the nominal voltage are enforced as bounds on each element of the diagonal voltage squares.

\[
V^2 \leq \text{diag}(v_i) \leq V_i^2 \quad \forall \ i \in \mathcal{N}^+ 
\quad (13)
\]

III. OPTIMIZATION PROBLEM

A. Market Optimization

The DSO will run the following day-ahead optimization problem \((|T| = 24)\) with an objective of minimizing the active- and reactive-power generation costs of all market participants.

\[
\begin{aligned}
\min_{v_0, \ell_1, S_{\ell_1}, s_{\ell_1}, \sigma} \quad & f = \sum_{t \in T} \sum_{\phi \in \Phi_t} \left( \sigma_s^P \text{real}(S_{\ell_1}^\phi) + \sigma_s^Q \text{imag}(S_{\ell_1}^\phi) \right) \\
& + \sum_{i \in \mathcal{N}_t} \sigma_s^P \text{real}(s_{\ell_1}^\phi) + \sigma_s^Q \text{imag}(s_{\ell_1}^\phi) \\
& + \sum_{i \in \mathcal{N}_s} \sigma_{svc} \text{imag}(s_{\ell_1}^\phi) \\
\text{s.t.} \quad & v_0 = V_0 V_0^H \\
(2), (5), (7), (8), (9), (12), (13)
\end{aligned}
\]

where \(\sigma\) denotes the generation bidding price with superscripts \(P\) and \(Q\) for active and reactive power; respectively, whilst \(S_{\ell_1}^1\) is the power flow from the substation.

B. Convexification

The optimization problem in (14) is nonconvex because of the rank constraint. However, by removing the constraint, an SDP-relaxed problem is obtained. Although the relaxation enlarges the feasible set, it is deemed tight only if \(X(v_i^\rho, \ell_1^t, S_{\ell_1}^\alpha)\) is rank one for every \(i \in \mathcal{N}^+\). In [7], it has been shown that a tight relaxation holds for most IEEE distribution feeders. For validation, a tightness check will be conducted for the case studies in this paper.

IV. DISTRIBUTION LOCALATIONAL MARGINAL PRICING

The resultant convex problem with linear equality and inequality constraints justifies the use of Lagrange multipliers to develop DLMPs. The Lagrangian function of the general form [9] for the overall single-period problem, with emphasis on the power balance equation, is

\[
\mathcal{L} = f(x) + \sum_{i \in \mathcal{F}} \lambda_i^P f_i(x) + \sum_{i \in \mathcal{M}} \mu_i h_i(x) \\
+ \sum_{i \in \mathcal{N}^+} \sum_{\phi \in \Phi_i} \lambda_i^\phi \text{real} \left( \sum_{k \in \Omega(i)} S_k^\phi - s_i^\phi - S_{\ell_1}^\phi + (z_i \ell_i)^\phi \right) \\
+ \sum_{i \in \mathcal{N}^+} \sum_{\phi \in \Phi_i} \lambda_i^\phi \text{imag} \left( \sum_{k \in \Omega(i)} S_k^\phi - s_i^\phi - S_{\ell_1}^\phi + (z_i \ell_i)^\phi \right)
\]

where \(f_i(x)\) represents the rest of equality constraints, and \(h_i(x)\) represents inequality constraints \((h_i(x) \leq 0)\). Thus, the partial derivative of (15) w.r.t real \((s_{\ell_1}^L)\) and imag \((s_{\ell_1}^L)\) yield

\[
\begin{aligned}
\text{A-DLMP} &= \lambda_i^P = \frac{\mathcal{L}}{\text{real}(s_{\ell_1}^L)} \quad \text{R-DLMP} &= \lambda_i^Q = \frac{\mathcal{L}}{\text{imag}(s_{\ell_1}^L)}
\end{aligned}
\]

V. CASE STUDIES

Numerical case studies are presented to quantify the DLMPs at each phase of an unbalanced distribution system. The SDP-based optimization problem is implemented using CVX toolbox [10] with MATLAB R2016b, and solved by Mosek solver [11].

A. Modified IEEE 37-bus Feeder

Fig. 2 illustrates the IEEE 37-bus feeder, an actual Californian feeder with unbalanced loading and line parameters at
each phase. The rated (peak) demand is 2.735 MVA with 0.9 PF lagging. All lines are three-phase configured, green buses are loaded. For Var-compensation, an SVC is installed at bus 28, with $\varphi = q = 150$ kVar, which as well offsets the voltages of neighboring buses. Also, three PVs are installed at bus 4, 33, and the loaded bus 18, while two wind turbines (WTs) are at bus 8 and 24. The transformer and regulators are ignored. The normalized demand, PV and wind profile multipliers in Fig. 3 are obtained from California ISO [12], and assumed as the adopted feeder’s day-ahead forecast available to the DSO. The net demand is based on 80% penetration of total DGs. Also, to facilitate the comparison of DLMPs among the three phases, the scale for $$/kWh or $$/kVarh is unified.

Case 1: Baseline

For the baseline case, the system runs as a passive distribution network, and only includes the SVC ($\sigma^P = \sigma^Q = 0$). Generation bidding prices of substation and SVC are $\sigma^P = 15$/kWh and $\sigma^Q = \sigma_{svc} = 3$/kVarh. Fig. 4 depicts the A-DLMPs and R-DLMPs.

The substation is the sole source of active power in this case, and so the A-DLMPs are governed by its generation cost. Fig. 4a-4c show the dynamic changes in A-DLMPs throughout the day. Their shared attribute is that they gradually increase as buses locate farther from the substation. This reflects the cost of not only generating the power, but also delivering it, involving the compensation of all real losses in the system. Therefore, in order for the substation to bring the system to a balance, it has to compensate for all real losses, which is the only offset to the generation cost. This is primarily true because the substation power is not constrained, and that all nodal voltages are above the minimum bound because of the SVC. As a result, the marginal congestion prices for A-DLMPs are zero. As for the differences, A-DLMPs among the three phases differ notably because of the unbalanced loads and line impedances (losses). Solid lines in Fig. 5 pictures A-DLMPs at 13:00, when network loading is 85%. Because phase C constitutes the largest portion of demand (44.4% of the total load), the DLMPs thereupon are the highest at most buses. Phase B has the smallest load, but its load at bus 10 is large, which explains its high A-DLMP compared to other phases. R-DLMPs increase in a similar trend except at buses near the SVC at bus 28. The SVC compensates for both reactive-power demand and losses, particularly those of buses/lines between 28, 29, and 30. At 13:00, the SVC injects 138.7, 72.2 and 150 kVar at phase A, B, and C. As seen in Fig. 5 it manages to bring R-DLMP at phase A and B to 3$/kVarh at the bus of installation and the neighboring buses. However, the large reactive-power load on phase C induces marginal congestion prices as their corresponding compensations bind the upper bound, and its R-DLMP is $3.0233$/kVarh. Therefore, the SVC is unable to serve the next demand increment at phase C, and thus R-DLMPs increases accordingly, reaching as high as $3.0931$/kVarh at peak demand (see Fig. 4f).

In simple terms, the highest A-DLMP (R-DLMP) possible in this case would be at a bus that is heavily loaded and remotely located from the substation (and SVC). In contemporary distribution networks, DLMP signals can be used to incentivize end-users and owners of spatially-installed DGs to either generate their power locally, or react with price-
sensitive demand response (DR) implementations. The former is featured in the next case study.

**Case 2: Inclusion of DGs**

In this case, PV- and wind-based DGs participate in the market with the same bidding prices as the substation. The penetration for the total PVs is 40% ($\omega=0.4$), and 0.9 PF leading/lagging for their inverters. The same percentage and PF are allocated for the wind turbines.

Fig. 6 shows a substantial change in DLMPs with the participation of DGs, mostly at times when DGs produce excessive power, i.e. PV peak. A lower net demand is viewed as light loading, and thereby it cuts down on DLMPs. Therefore, the shadow price in this case is interpreted as the per-unit change of net demand in the optimal cost that is obtained by relaxing or strengthening the power-balance constraints. In other words, it is the marginal price saved by relaxing the constraint (increasing the penetration), or the marginal cost added by strengthening the constraint (decreasing the penetration). Fig. 7 manifests the change in marginal prices at 13:00, where $\Delta P$ and $\Delta Q$ are the difference of A-DLMPs and R-DLMPs between the baseline case and this case. The change is the DG contribution to reducing losses, especially those of remote lines, and alleviating the SVC binding cost.

Active power generation output and A-DLMPs at the bus of installation are further investigated at 13:00, and listed in Table I with shaded cells indicating congested generations. The DG generations are burdened at phase A and C because of the phase heavy loading, resulting in an increase of their marginal congestion costs, predominantly at distant buses. The light active power loading in phase B, however, mostly curtails its output so as to balance supply with demand, except for the large demand at bus 10 which burdens WT8 and adds congestion costs. The incentive is that increased penetration is a good alternative to alleviate the congestion and enable DGs to serve the next increment of demand.

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**TABLE I: DG Active-power Output and A-DLMPs at 13:00**

<table>
<thead>
<tr>
<th>Bus</th>
<th>A kW $/kWh</th>
<th>B kW $/kWh</th>
<th>C kW $/kWh</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub</td>
<td>144.9</td>
<td>71.3</td>
<td>371</td>
</tr>
<tr>
<td>PV4</td>
<td>109.2</td>
<td>103</td>
<td>109.2</td>
</tr>
<tr>
<td>PV18</td>
<td>109.2</td>
<td>76.7</td>
<td>109.2</td>
</tr>
<tr>
<td>PV33</td>
<td>109.2</td>
<td>54.7</td>
<td>109.2</td>
</tr>
<tr>
<td>WT8</td>
<td>8.8</td>
<td>141.3</td>
<td>94.9</td>
</tr>
<tr>
<td>WT24</td>
<td>141.3</td>
<td>100</td>
<td>141.3</td>
</tr>
</tbody>
</table>

The A-DLMPs at phase B of PV4 and PV18 mark an interesting case that indicates a slightly-excessive generation causes prices to drop well below the generation cost. Thus, the A-DLMP signals should incentivize the corresponding PV owners to lower their generation.

**Case 3: Lower DG Bidding Prices**

In the previous case, identical bidding prices are assumed, and so loads are fed by the nearest source of energy. It is expected that if DGs bid with lower prices, the optimization problem would prioritize their generation over the substation (and SVC), which impacts the DLMPs accordingly.
Setting $\sigma_p^P = \$13$/kWh and $\sigma_q^Q = \$2$/kVarh and the same penetration level and PF as in Case 2, Fig. 8 shows that DLMPs are reduced in general, and are minimum at buses of DG installation reaching as low as $\$14.6763$/kWh and $\$2.8384$/kVarh at phase A of WT8 bus. DG generations and A-DLMPs in Table II show that all DGs are bound by their maximum limit because of their lower generation cost. This prompts the DG owners to increase their generation capacity to participate in imbalance and loss reduction for the next increment of load, and alleviate the existing congestion costs.

Fig. 9 shows the decay of the total consumer’s payment for the entire day with respect to incremental DG penetration, where $\Delta P$ and $\Delta S$ are the sum of the payments at all phases, buses and time horizons.

B. Tightness of SDP Relaxation

The solution is said to be tight if the matrix solutions $(\sum_{i \in T} \sum_{i \in N^+} X_{i,i}^t)$ are rank one. The rank is examined by computing the division of the second largest eigenvalue by the largest eigenvalue, $|\text{eig}_2/\text{eig}_1|$, where $\text{eig}_2 > \text{eig}_1 > 0$. Smaller ratios indicate the solution proximity to being rank one. The maximum ratio over all matrices and time horizons is computed in (17) for every case and listed in Table III.

\[
\text{Tightness} = \max \left(\sum_{t \in T} \sum_{i \in N^+} |\text{eig}_2/\text{eig}_1| \right)
\]

Table III: Tightness of Numerical Solutions

<table>
<thead>
<tr>
<th>Case</th>
<th>Tightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.1615e-07</td>
</tr>
<tr>
<td>2</td>
<td>6.1587e-07</td>
</tr>
<tr>
<td>3</td>
<td>4.9779e-07</td>
</tr>
</tbody>
</table>

Since the overall solution ratios satisfy sufficiently small values, $|\text{eig}_2/\text{eig}_1| \leq 6.1587 \times 10^{-7}$, the SDP relaxation is tight.

VI. CONCLUSION

This paper utilizes a convex AC OPF formulation to develop DLMP for active and reactive power references in a multiphase distribution networks. As a result of the unbalanced loads, DLMPs differ per-phase. In passive distribution networks, results substantiate that buses farther from substation and Var-compensation devices incur higher DLMPs corresponding to marginal loss prices. On the other hand, when renewable-energy distributed generators engage in the distribution market, overall DLMPs decrease proportionally with the penetration level, highlighting the financial benefits for the new market participants, and the potential contribution to the balance and loss reduction in the entire system.

REFERENCES