Mixed Integer Linear Programming and Nonlinear Programming for Optimal PMU Placement

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***Abstract*— Phasor measurement units (PMUs) provide synchronized measurements of voltage and current phasors and can make state estimation more accurate. The objective of optimal PMU placement (OPP) problem is to minimize the number of PMUs required for the system to be completely observable. This paper presents two different formulations of optimal PMU placement (OPP) problem: mixed integer linear programming (MILP) and nonlinear programming (NLP). For each formulation, modeling of power flow measurements, zero injection, limited communication facility, and single PMU failure is studied. The contribution of our paper is to conduct a comparison between the MILP and NLP formulations and show the advantages and disadvantages of each formulation.**

***Index Terms*— Mixed integer linear programming, nonlinear programming, optimal PMU placement**

I. INTRODUCTION

The power system security needs to have a detailed monitor to the operating conditions of the system. In the control center, the state estimator can deal with the measurements received from the remote terminal units (RTUs) at the substations. Those measurements include bus voltages, branch currents, real and reactive power flows, and power injections. Recently, phasor measurement units (PMUs) with the help of global positioning system (GPS) can provide synchronized phasor measurements of voltages and currents [1]. That will improve the performance of the state estimator because the synchronized measurements provided by PMUs during the dynamic event will make the state estimation more accurate [2]. When a PMU is placed at one bus, it can measure the bus phasor voltage and phasor currents of all lines connected to that bus making the system observable [3]. However, integrating PMUs at each bus instantaneously is difficult and expensive. Then the problem that needs to be solved is to find the optimal PMU placement that can make the system observable using PMUs at certain buses.

In this paper, the optimal PMU placement (OPP) problem is solved using two different approaches which are mixed integer linear programming (MILP) and nonlinear programming (NLP). The system is then studied over several cases including power flow measurements, zero injection, limited communication facility, and single PMU failure. A comparison between the two approaches is conducted to show their advantages and disadvantages. In addition, system observability redundancy index (SORI) is demonstrated to provide a higher redundancy. Therefore, the objective of this formulation is to minimize the total number of PMUs and maximize the redundancy measurements in power systems.

The remaining sections are organized as follows. Section II and section III explain mixed integer linear programming formulation and nonlinear programming formulation. Section IV investigates the aforementioned four case studies. Section V and VI are OPP problem simulation results and conclusion.

II. MIXED INTEGER LINEAR PROGRAMMING FORMULATION

The power system state estimation investigated in this paper is a dc power flow based linear state estimation.

$$z = Hx + e \tag{1}$$

where z is the measurement vector, x is the state vector, H is the measurement matrix, and e is the error vector. The states are the voltage phase angle (θ_i for Bus i) of every bus in the power grid. The measurements are assumed to come from PMUs, which include phase angle of Bus i and the power flow measurements from Bus i to adjacent buses ($j \in ad_i$, where ad_i is the buses adjacent to Bus i). Therefore, if θ_i is measured by the PMU, then θ_j can also be found since P_{ij} is measurable ($P_{ij} = \frac{1}{X_{ij}}(\theta_i - \theta_j)$ where X_{ij} is the reactance of the line). Thus, with a PMU installed at Bus i, Bus i and all its adjacent buses are observable.

The following is a mixed integer linear programming formulation [3]:

$$\min_{x} \quad \sum_{k=1}^{N} w_k \; x_k \tag{2a}$$

$$\text{subject to:} \quad Ax \geq B \tag{2b}$$

$$x_i \in \{0, 1\}, i = 1, \cdots, N \tag{2c}$$

where x_i notates if Bus i has a PMU installed ($x_i = 1$) or not ($x_i = 0$), and the objective function is the minimization of the number of PMUs to be installed. The w_k is the PMU installation cost. Note that all PMUs have the same installation cost ($w_i = 1$) which makes the minimization of the PMU installation cost is equivalent to the minimization of the number of PMUs. The A matrix's element is as follows.

$$a(i,j) = \begin{cases} 1, \text{ if Bus } i \text{ and Bus } j \text{ are connected} \\ 1, i = j \\ 0. \text{ if Bus } i \text{ and Bus } j \text{ are not connected} \end{cases}$$

The B matrix will be:

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$$

Take the ith constraint: $(A)_i x \geq 1$ (where $_i$ means ith row). It means that at least one PMU should be placed on one of the buses that are connected to Bus i or on Bus i.

To explain the above formulation, the mixed integer linear programming problem is solved for the IEEE 14-bus system as shown in Fig. 1.

Based on the problem formulation, the mixed integer linear programming will be as the following:

$$\begin{aligned} \min_{x} \quad & \sum_{k=1}^{14} x_k \\ \text{subject to:} \quad & Ax \geq B \\ & x_i \in \{0,1\}, i = 1, 2, \cdots 14. \end{aligned}$$

Then we can find matrix A based on the $a(i,j)$ entries as the following:

$$A = \begin{bmatrix} 1\,1\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,1\,1\,1\,1\,0\,1\,0\,1\,0\,0\,0\,0\,0 \\ 1\,1\,0\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,0\,0\,0\,0\,1\,1\,1\,0 \\ 0\,0\,0\,1\,0\,0\,1\,1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,1\,1 \end{bmatrix}$$

The constraints can be formed as:

$$f(x) = \begin{cases} f_1 = x_1 + x_2 + x_5 \geq 1 \\ f_2 = x_1 + x_2 + x_3 + x_4 + x_5 \geq 1 \\ f_3 = x_2 + x_3 + x_4 \geq 1 \\ f_4 = x_2 + x_3 + x_4 + x_5 + x_7 + x_9 \geq 1 \\ f_5 = x_1 + x_2 + x_4 + x_5 + x_6 \geq 1 \\ f_6 = x_5 + x_6 + x_{11} + x_{12} + x_{13} \geq 1 \\ f_7 = x_4 + x_7 + x_8 + x_9 \geq 1 \\ f_8 = x_7 + x_8 \geq 1 \\ f_9 = x_4 + x_7 + x_9 + x_{10} + x_{14} \geq 1 \\ f_{10} = x_9 + x_{10} + x_{11} \geq 1 \\ f_{11} = x_6 + x_{10} + x_{11} \geq 1 \\ f_{12} = x_6 + x_{12} + x_{13} \geq 1 \\ f_{13} = x_6 + x_{12} + x_{13} + x_{14} \geq 1 \\ f_{14} = x_9 + x_{13} + x_{14} \geq 1 \end{cases}$$

III. Nonlinear Programming Formulation

A. *NLP Formulation*

In addition to the mixed integer linear programming, optimal PMU placement can be formulated as nonlinear programming which can be solved by sequential quadratic programming (SQP) [4], [5]. Nonlinear programming formulation can result in several solutions for the optimal PMU placement problem, whereas the mixed integer linear programming obtains only one solution.

In nonlinear programming formulation, x_i is no longer a binary variable. Rather, it is treated as a continuous variable. To enforce x_i to be either 1 or 0, the following constraint is used: $x_i(x_i - 1) = 0$.

The nonlinear programming formulation minimizes the quadratic objective function, which represents the total PMU installation cost, subject to nonlinear equality constraints. The lower bound and upper bound of the decision variables are 0 and 1 respectively. Note that the network observability is represented by the nonlinear equality constraints [4], [5].

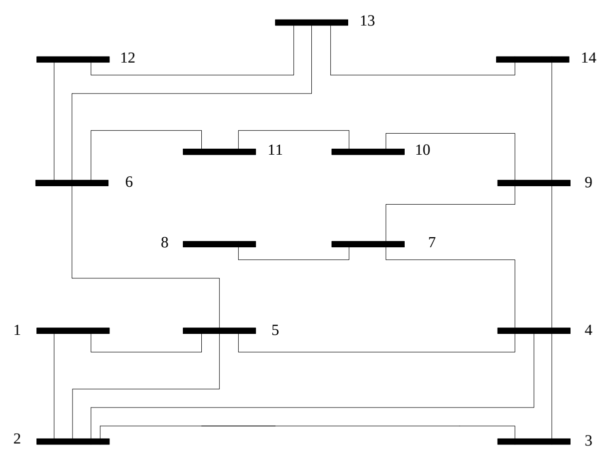

Fig. 1. IEEE 14-Bus System.
Reference: Power System Case Archive by University of Washington.

The following is a nonlinear programming formulation for optimal PMU placement:

$$\min_{x} \quad J(x) = x^T W x = \sum_{k=1}^{N} w_k \; x_k^2 \tag{3a}$$

$$\text{s.t.:} \quad g_i(x) = (\; 1 - x_i) \prod_{j \in ad_i} (\; 1 - x_j) \; = 0 \tag{3b}$$

$$0 \leq x_i \leq 1, \quad \text{for all } i \in \boldsymbol{S} \tag{3c}$$

where $J(x)$ is the objective function, x^T is the vector of the PMU placement, W is the diagonal weight (PMU installation cost) matrix, ad_i is the buses adjacent to Bus i, and $\boldsymbol{S}$ is the set of buses in the system.

This is a nonconvex problem due to the nonlinear constraints which lead to several local minimum points to the optimization problem [4]. This gives us different solutions for the optimal PMU placement shall we start from different initial x. This nonlinear programming is solved by sequential quadratic programming (SQP) algorithm.

To explain the above formulation, the nonlinear programming problem is solved for IEEE 14-bus system (Fig. 1), and it is compared with the solution of the mixed integer linear programming.

Based on IEEE 14-bus system, the nonlinear programming will be as the following:

$$\begin{aligned} &\min_{x} \quad \sum_{k=1}^{14} x_k^2 \\ &\text{s.t.:} \quad g_i(x) = (\ 1 - x_i) \prod_{j \in ad_i} (\ 1 - x_j) \ = 0 \\ &\qquad\quad 0 \leq x_i \leq 1, \quad i = 1, 2, \cdots 14. \end{aligned}$$

where $g_i(x)$ will be as the follows.

$$g_i(x) = \begin{cases} g_1 = (\ 1 - x_1)(\ 1 - x_2)(\ 1 - x_5) = 0 \\ g_2 = (\ 1 - x_2)(\ 1 - x_1)(\ 1 - x_3)(\ 1 - x_4) \\ (\ 1 - x_5) = 0 \\ g_3 = (\ 1 - x_3)(\ 1 - x_2)(\ 1 - x_4) = 0 \\ g_4 = (\ 1 - x_4)(\ 1 - x_2)(\ 1 - x_3)(\ 1 - x_5) \\ (\ 1 - x_7)(\ 1 - x_9) = 0 \\ g_5 = (\ 1 - x_5)(\ 1 - x_1)(\ 1 - x_2)(\ 1 - x_4) \\ (\ 1 - x_6) = 0 \\ g_6 = (\ 1 - x_6)(\ 1 - x_5)(\ 1 - x_{11})(\ 1 - x_{12}) \\ (\ 1 - x_{13}) = 0 \\ g_7 = (\ 1 - x_7)(\ 1 - x_4)(\ 1 - x_8)(\ 1 - x_9) = 0 \\ g_8 = (\ 1 - x_8)(\ 1 - x_7) = 0 \\ g_9 = (\ 1 - x_9)(\ 1 - x_4)(\ 1 - x_7)(\ 1 - x_{10}) \\ (\ 1 - x_{14}) = 0 \\ g_{10} = (\ 1 - x_{10})(\ 1 - x_9)(\ 1 - x_{11}) = 0 \\ g_{11} = (\ 1 - x_{11})(\ 1 - x_6)(\ 1 - x_{10}) = 0 \\ g_{12} = (\ 1 - x_{12})(\ 1 - x_6)(\ 1 - x_{13}) = 0 \\ g_{13} = (\ 1 - x_{13})(\ 1 - x_6)(\ 1 - x_{12}) \\ (\ 1 - x_{14}) = 0 \\ g_{14} = (\ 1 - x_{14})(\ 1 - x_9)(\ 1 - x_{13}) = 0 \end{cases}$$

Note that we assume that the installation cost (w_i) for all buses is equal to 1.

This problem can be solved using $fmincon$ function in MATLAB which can solve nonlinear programming (NLP) using sequential quadratic programming solver (SQP). Then the optimal PMU placement will be as the following:

$$x = \begin{bmatrix} 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0 \end{bmatrix}^T$$

Thus, the optimal PMU placement will be on buses 2, 8, 10, and 13.

The same problem is solved using the mixed integer linear programming by MATLAB and *intlinprog* function, and we found the same optimal placement which will be on buses 2, 8, 10, and 13.

As mentioned above, the nonlinear programming provides several solutions to the problem. Thus, we changed the initial values to be random numbers in the feasible set whose decision variable x should be larger than zero and less than one. After several iterations, we got the following optimal solutions:

$$x = \begin{bmatrix} 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0 \end{bmatrix}^T,$$

$$x = \begin{bmatrix} 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0 \end{bmatrix}^T,$$

$$x = \begin{bmatrix} 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}^T,\ and$$

$$x = \begin{bmatrix} 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}^T$$

Therefore, nonlinear programming is an effective way to find the optimal PMU placement providing several optimal solutions to choose from. Table I shows the comparison between mixed integer linear programming and nonlinear programming for the optimal PMU placement.

TABLE I. OPP RESULTS USING MILP AND NLP/SQP

Test Algorithm (IEEE 14-Bus)	Minimum Optimal Set	PMU Placement Bus	CPU Time (s)
MILP	4	2,8,10,13	0.06
NLP/SQP	4	2,8,10,13 2,7,10,13 2,7,11,13 2,6,7,9 2,6,8,9	0.7

B. System Observability Redundancy Index

In order to maximize the redundancy measurements in the power system, the system observability redundancy index (SORI) should be considered. SORI is an essential parameter for the security monitoring of the power system [6]. SORI is defined as the total number of all Bus Observability Index (BOI) of the system, where BOI is the number of PMUs that can observe a certain bus. Then SORI is given by the following [6], [7]:

$$\sigma = \sum_{i=1}^{k} \ \beta_i \tag{4}$$

where σ is SORI, and β_i is BOI for a certain bus i.

SORI gives the ability to choose the perfect minimum optimal set among the other sets [6], [7]. It can make the power system more reliable providing the maximum redundancy measurements. When the number of SORI is high, the optimal set presents higher measurement redundancy. Table II presents different optimal sets with their SORI. Therefore, maximum redundancy measurements will be provided in the power system.

TABLE II. SORI FOR EACH OPTIMAL SET

Minimum Optimal Set	PMU Placement Bus	SORI
4	2,8,10,13	14
	2,7,10,13	16
	2,7,11,13	16
	2,6,8,9	17
	2,6,7,9	19

Thus, the objective of the optimal PMU placement formulation is to minimize the total number of the PMUs and maximize the redundancy measurements in the power system.

IV. OPTIMAL PMU PLACEMENT CASE STUDIES

A. OPP Formulation with Power Flow Measurements

Let's assume that there is a power flow meter on Line ij in the system. When either Bus i's or Bus j's state (phase angle) is known, other bus's angle can be found should $P_{ij} = \frac{1}{X_{ij}}(\theta_i - \theta_j)$ (where X_{ij} is the reactance of the line) is given.

1) MILP Formulation: The observability constraints should be changed considering power flow measurements. In the absence of the power measurements on lines i–j, the observability constraints of the two buses will be given by:

$$f_i(x) = (A)_i x \geq 1 \tag{5a}$$

$$f_j(x) = (A)_j x \geq 1 \tag{5b}$$

where $(A)_i$ means the ith row of A matrix, and $(A)_j$ means the jth row of A matrix. Then in the presence of the power measurements on lines i–j, the above two constraints will be merged into the following joint constraint [3], [8]:

$$f_{flow,i} = f_i(x) + f_j(x) \geq 1 \tag{6}$$

The above constraint means that as long as Bus i or Bus j is observable, then the other bus is also observable due to the power flow meter.

2) NLP Formulation: The observability constraints of the NLP formulation should be changed considering the power flow measurements as well. Then the two constraints of Bus i and Bus j in the absence of power flow measurement will be merged into a joint constraint in case of flow measurements on lines i–j as follows [6].

$$g_i(x) = 0 \tag{7a}$$

$$g_j(x) = 0 \tag{7b}$$

$$g_{flow,i}(x) = g_i(x) g_j(x) = 0 \tag{8}$$

Note that a high order term $(1 - x_i)^n$ can be generated because of the common existence of the term $(\ 1 - x_i)$ of the two observability constraints $g_i(x) = 0$ and $g_j(x) = 0$ [6]. Since the right hand of the observability constraint is zero, the high order term will be equivalent to the first order term.

3) Example of OPP Formulation with Power Flow Measurements: Let's assume that there are power flow measurements on lines 2-3, 3-4, 6-11, 6-12, and 7-8 in the IEEE 14-bus system (Fig. 1). Then we are going to find $f_{flow,i} \geq 1$ and $g_{flow,i} = 0$ for MILP and NLP respectively.

a) MILP Joint Constraints: First, the power flow measurements on lines 2-3 and 3-4:

$$f_2 = x_1 + x_2 + x_3 + x_4 + x_5 \geq 1$$
$$f_3 = x_2 + x_3 + x_4 \geq 1$$
$$f_4 = x_2 + x_3 + x_4 + x_5 + x_7 + x_9 \geq 1$$

Finding the joint constraint from (6) as follows.

$$\begin{aligned} f_{flow,2} &= f_2 + f_3 + f_4 \geq 1 \\ &= x_1 + 3x_2 + 3x_3 + 3x_4 + 2x_5 + x_7 + x_9 \geq 1 \end{aligned}$$

The joint constraint indicates that whenever one of the buses (2,3,4) is observable, the rest are observable due to the meters. Thus, we merged the constraints f_2, f_3, and f_4 into one joint constraint $f_{flow,2}$ to assure a placement of one PMU for one of those buses or their adjacent buses. It is obvious that in the case of the power flow measurements, the number of PMUs will be reduced (see Table III).

Second, the power flow measurements on lines 6-11 and 6-12:

$$f_6 = x_5 + x_6 + x_{11} + x_{12} + x_{13} \geq 1$$
$$f_{11} = x_6 + x_{10} + x_{11} \geq 1$$
$$f_{12} = x_6 + x_{12} + x_{13} \geq 1$$

The joint constraint will be:

$$f_{flow,6} = x_5 + 3x_6 + x_{10} + 2x_{11} + 2x_{12} + 2x_{13} \geq 1$$

Third, the power flow measurement on line 7-8 will result in the following:

$$f_{flow,7} = x_4 + 2x_7 + 2x_8 + x_9 \geq 1$$

b) NLP Joint Constraints: The NLP constraints for buses $2, 3$, and 4 will be as the following:

$$g_2 = (\ 1 - x_2)(\ 1 - x_1)(\ 1 - x_3)(\ 1 - x_4)(\ 1 - x_5) = 0$$
$$g_3 = (\ 1 - x_3)(\ 1 - x_2)(\ 1 - x_4) = 0$$
$$\begin{aligned} g_4 = &(\ 1 - x_4)(\ 1 - x_2)(\ 1 - x_3)(\ 1 - x_5)(\ 1 - x_7) \\ &(\ 1 - x_9) = 0 \end{aligned}$$

The joint constraint for the NLP formulation considering the power flow measurements on lines 2-3 and 3-4 which is obtained from (8) will be as the following:

$$g_{flow,2} = g_2 g_3 g_4 = 0$$
$$= (1-x_1)(1-x_2)(1-x_3)(1-x_4)(1-x_5)$$
$$(1-x_7)(1-x_9) = 0$$

The joint constraints for power flow measurements on lines 6-11, 6-12, and 7-8 will be as follows.

$$g_{flow,6} = g_6 g_{11} g_{12} = (1-x_5)(1-x_6)(1-x_{10})$$
$$(1-x_{11})(1-x_{12})(1-x_{13}) = 0$$

$$g_{flow,7} = g_7 g_8 = (1-x_4)(1-x_7)(1-x_8)(1-x_9) = 0$$

B. OPP Formulation with Zero Injection

1) MILP and NLP Formulation: Supposing that the system has a zero injection pseudo measurement at Bus i, then we obtain a constraint considering that the power injection at Bus i is zero. The observability constraints of the problem that account for the injection bus are incorporated into a single joint constraint for MILP as follows [3].

$$f_{injt,i}(x) = f_i(x) + \sum_{j \in ad_i} f_j(x) \geq m - 1 \tag{9}$$

where m is the total number of Bus i and its adjacent buses.

Therefore, the pseudo measurement of the zero injection bus leads to calculate a voltage phasor for one bus from its adjacent buses as long as voltage phasors of all remaining buses are available [9]. The joint observability constraint provides one constraint for the observability constraints of the injection Bus i and all of the adjacent buses to that bus (ad_i). There is no difference between the zero and nonzero injection bus [8].

The NLP joint constraint is developed in [6], and expressed as the following:

$$g_{injt,i}(x) = g_i(x) \prod_{j \in ad_i} g_j(x) = 0 \tag{10}$$

This joint constraint is not equivalent to (9). The joint constraint (10) indicates that Bus i and its adjacent buses are observable whenever one of those buses is observable which may end up with unobservable buses.

2) Example of OPP Formulation with Zero Injection Bus: Let's assume that Bus 7 is a zero injection bus (ZIB) in the IEEE 14-bus system (Fig. 1). Then $f_{injt,7} \geq m - 1$ and $g_{injt,7} = 0$ are found to show how these two constraints are not equivalent.

a) MILP and NLP Joint Constraint: The adjacent buses to Bus 7 are buses 4, 8, and 9 ($m = 3$). Then we are going to find the joint constraint for MILP as the following:

$$f_4 = x_2 + x_3 + x_4 + x_5 + x_7 + x_9 \geq 1$$
$$f_7 = x_4 + x_7 + x_8 + x_9 \geq 1$$
$$f_8 = x_7 + x_8 \geq 1$$
$$f_9 = x_4 + x_7 + x_9 + x_{10} + x_{14} \geq 1$$

Finding the MILP joint constraint from (9) as follows.

$$f_{injt,7} = f_4 + f_7 + f_8 + f_9 \geq 3$$
$$= x_2 + x_3 + 3x_4 + x_5 + 4x_7 + 2x_8 + 3x_9 + x_{10}$$
$$+ x_{14} \geq 3$$

While the NLP joint constraint is obtained from (10) as the following:

$$g_{injt,7} = g_4 g_7 g_8 g_9 = 0$$
$$= (1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_7)$$
$$(1-x_8)(1-x_9)(1-x_{10})(1-x_{14}) = 0$$

The MILP joint constraint means that if three of the buses (4,7,8,9) are observable, the fourth bus will be observable. On the other hand, it is clear that the NLP joint constraint means that if one of buses (4,7,8,9) is observable, the other three buses will be observable which might result in a wrong solution.

C. OPP Formulation with Limited Communication Facility

The limited communication facility in the substation can prevent the PMU installation due to the lack of data links required to enable the communication between PMUs and the control center. This problem can affect the installation cost of the PMU to be much higher [10]. Thus, a high installation cost w_i will be assigned to the bus that has a limited communication facility for both MILP and NLP. Consequently, the high installation cost will exclude the limited communication facility buses from the optimal set [6].

Let's assume that there are limited communication facilities at buses 2 and 9 on the IEEE 14-bus system (Fig. 1). Then high installation costs ($w_i = 10^9$) are assigned to buses 2 and 9, whereas installation costs of the other buses are kept as $w_i = 1$.

D. OPP Formulation with Single PMU Failure

Although PMUs are reliable devices, failure of a single PMU is possible. Therefore, to protect the system from losing one PMU and leaving the system unobservable, the optimal PMU set is divided into two sets which are main set and backup set. The main set is the set obtained without a PMU failure, while the backup set is the set that we are going to obtain. For MILP formulation, every bus in the system is going to be observed by two PMUs which modify the right hand side of the inequality constraints to be two instead of one [7]. Also, the backup set of the MILP formulation can be obtained by removing x_i terms that represent the main set. Similarly, the backup set for the NLP formulation is going to be obtained by removing all of the terms $(1-x_i)$ and $(1-x_j)$ that are related to the main set from the observability constraints [6]. This will ensure that the same bus will not pick up more than one time. Thus, the backup set will keep the system observable in case of a single PMU failure.

Note that we get the main set by solving the problem without considering the PMU failure, and we choose the main set to be on buses 2, 8, 10, and 13 (see Table I). Thus, we removed all of the terms x_2, x_8, x_{10}, and x_{13} from the MILP constraints. Likewise, we removed all of the terms $(1 - x_i)$ and $(1 - x_j)$ from the observability constraints of NLP which are $(1 - x_2)$, $(1 - x_8)$, $(1 - x_{10})$, and $(1 - x_{13})$ because they are related to the main set.

TABLE III. A COMPARISON BETWEEN NLP AND MILP

Optimal PMU Placement Problem Case	Minimum Optimal Set	MILP PMU Placement	NLP/SQP PMU Placement
Power flow measurements (on lines 2-3, 3-4, 6-11, 7-8, and 6-12)	3	1,7,13	1,4,13 1,6,9 **1,7,13** 1,9,12 2,4,13 2,6,9 2,7,13 2,9,13 2,9,12 5,7,13 5,9,12
Limited communication facility (on buses 2 and 9)	5	3,5,8,10,13	1,3,7,10,13 1,3,7,11,13 1,3,8,10,13 1,4,7,10,13 1,4,8,10,13 3,5,7,11,13 **3,5,8,10,13** 4,5,7,10,13 4,5,7,11,13 4,5,8,10,13 4,5,8,11,13
Single PMU failure	9	2,8,10,13 (Main) + 1,4,6,7,9 (Backup)	2,8,10,13 (Main)+ 1,3,6,7,9 (Backup) **2,8,10,13 (Main)+ 1,4,6,7,9 (Backup)** 2,8,10,13 (Main)+ 3,5,6,7,9 (Backup) 2,8,10,13 (Main)+ 4,5,6,7,9 (Backup)

V. OPP PROBLEM SIMULATION RESULTS

A comparison between NLP and MILP is conducted for the aforementioned OPP cases. Power flow measurements, limited communication facility, and single PMU failure are formulated using mixed integer linear programming and non-linear programming. The optimization problems are solved by MATLAB *intlinprog* function for MILP and *fmincon* function using sequential quadratic programming solver (SQP) for NLP. Table III presents the comparison between NLP and MILP. It is clear that MILP solution matches with one of the optimal sets provided by NLP. In the power flow measurements case, the number of PMUs in both formulations is reduced to be 3 instead of 4 in the general case (see Table I) due to the power flow meter. In contrast, the number of PMUs is increased in the limited communication facility case as shown in Table III. In the single PMU failure case, another optimal set is provided to be a backup set which would be very expensive to install. Note that we exclude the zero injection case from the comparison results since the nonlinear joint constraint for zero injection ends up with a wrong solution as explained in section IV-B.

Both MILP and NLP are effective ways to solve the OPP problem, and each formulation has its advantages and disadvantages. It is obvious that MILP formulation has less computational time compared to NLP formulation as can be seen from Table I. On the other hand, NLP formulation can provide more than one optimal solution with the same cost to the OPP problem.

VI. CONCLUSION

Mixed Integer linear programming formulation for the OPP problem is presented for complete observability. Non-linear programming formulation for the OPP problem is demonstrated using sequential quadratic programming. NLP can obtain more than one optimal solution to choose from which is an advantage over the MILP formulation. Moreover, the sequential quadratic programming is used for the OPP problem to minimize the number of PMUs as well as maximize the redundancy measurements in the power system. Several observability contingencies, which are power flow measurements, zero injection, limited communication facility, and single PMU failure, are discussed for both approaches. The advantages and disadvantages of the two formulations are presented.

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