

Tutorial on cutting plane methods for economic dispatch problems

Lingling Fan

Department of Electrical Engineering

University of South Florida

Tampa, FL 33620

url: <http://power.eng.usf.edu>

Email: linglingfan@usf.edu

Abstract—In this article, three economic dispatch examples will be given to show the philosophy and general steps on cutting plane methods (Bender’s decomposition and Branch and Bound (B&B) algorithm). Bender’s decomposition employs duality to generate cuts, while B&B employs linear programming (LP) relaxation to find lower bound for a minimization problem and generate subregions. We will show a continuous variable problem solved by Bender’s decomposition and then show a mixed integer programming (MIP) problem solved by both Bender’s decomposition and B&B algorithm. In addition, demo codes in CVX will be given. This article also serves as an example how to tackle a new area and how to learn a new algorithm or method. Our approach is learning by simple examples.

I. INTRODUCTION

This article serves as an example how to tackle a new area and how to learn a new algorithm or method. Our approach is learning by simple examples. Here we will show how to understand cutting plane methods used in solving comprehensive optimization problems. Our approach is learning by simple examples. This approach will be demonstrated in three examples.

A cutting plane method, in short, is to generate a reduced feasible region through cuts. A well-known example is Bender’s decomposition where a problem is converted to a master problem and a subproblem. The master problem deals with only integer variables and the subproblem deals with only continuous variables. When the integer variables are fixed, the subproblem can be solved with ease. Through Lagrangian dual, a cut will be generated by the subproblem.

Since there are commercial toolboxes available to handle convex programming problems and mixed linear integer programming problems, why do we still need cutting plane methods? There are two types of scenarios requiring this type of methods. One is large-scale scenario. The set of the decision variables has a large dimension. This set can be separated into subsets. With one subset fixed, the other subset can be found and a cut will be generated. Iteratively, a large-scale optimization problem can be solved. This is especially true when integer variables are in the picture. For example, CVX toolbox [1] has a limit of handling variables. When the size of a problem increases, CVX will keep running without giving a solution. Stochastic programming problems usually deal with a large set of scenarios with a large-scale dimension.

Therefore, Bender’s decomposition has been seen in stochastic programming problem solving [2].

One can argue that distributed optimization such as dual decomposition, primal decomposition can also be used to solve large-scale problem iteratively. However, primal or dual or ADMM guarantee convergence for convex programming problems only. MIP problems are nonconvex. In that sense, cutting plane methods are more powerful since they can handle mixed integer programming problems.

The second type of problems can not be solved by commercial solvers such as CVX and CPLEX. Take a look at nonlinear mixed integer programming problems. For example, select the best locations for m capacitors in a AC network. This type of problems cannot be solved by CVX or CPLEX. If the problem is continuous, at least the local optimization point can be found by carrying out interior point method. However, when there are mixed integer variables, it is very difficult to handle these problems. Cutting plane methods provide a way to solve this type of problems.

In the following two sections, we will use two very simple economic dispatch examples to explain Bender’s decomposition. We also demonstrate in the third example Branch and Bound to solve the same MIP problem.

Of course, there are tutorials around on Bender’s decomposition, e.g., [3]. The benefit of this tutorial is the simplicity of the examples as well as the demo code provided. That way, we believe readers can learn in an expedite way.

II. EXAMPLE 1: CONTINUOUS PROBLEM SOLVED BY BENDER’S DECOMPOSITION

In the first example, we look at a simple economic dispatch problem: two generators are serving loads. The size of the load is 11 MW. The two generators’ output power levels are limited to be 10 MW and 5 MW.

$$\min_{P_1, P_2} 2P_1^2 + 4P_2^2 \quad (1)$$

$$\text{s.t. } P_1 + P_2 = 11 \quad (2)$$

$$0 \leq P_1 \leq 10 \quad (3)$$

$$0 \leq P_2 \leq 5 \quad (4)$$

To solve it by the cutting plane method, we first examine the problem and convert the problem to be a two-level problem.

$$\min_{P_1} 2P_1^2 + \{\min_{P_2} 4P_2^2 | P_2 = 11 - P_1; 0 \leq P_2 \leq 5\} \quad (5)$$

$$\text{s.t. } 0 \leq P_1 \leq 10 \quad (6)$$

$$0 \leq 11 - P_1 \leq 5 \quad (7)$$

Note that the feasible region of P_1 in the master problem has been modified. The two inequality constraints can be combined as $6 \leq P_1 \leq 10$.

If P_1 is fixed, the subproblem $\{\min_{P_2} 4P_2^2 | P_2 = 11 - P_1; 0 \leq P_2 \leq 5\}$ can be solved. In addition, through this step, a Lagrangian dual based cut will be generated. The dual problem of the subproblem is

$$\max_{\lambda} \min_{P_2} 4P_2^2 + \lambda(11 - P_1 - P_2) \quad (8)$$

$$\text{s.t. } 0 \leq P_2 \leq 5 \quad (9)$$

For a given $P_1(k)$ at k -step, the corresponding $\lambda(k)$ and $P_2(k)$ can be found by solving the subproblem. Based on the dual problem, the subproblem is to find the maximum point. Solution at $\lambda(k)$ is a lower bound for the objective function. Therefore, a cut can be introduced to form a master problem.

$$\min_{P_1} 2P_1^2 + \eta \quad (10)$$

$$\text{s.t. } \eta \geq 4P_2^2(k) + \lambda(k)(11 - P_1 - P_2(k)) \quad (11)$$

$$6 \leq P_1 \leq 10 \quad (12)$$

Essentially, the dual cut generated is nothing but a cut following the logic presented in S. Boyd's notes of EE 364b [4]. Consider an unconstrained problem

$$\text{minimize } f_0(x), \quad (13)$$

where f_0 is convex. To find a cutting-plane for this problem, at point x , we can generate a deep cut

$$\eta \geq f_0(x) + g^T(z - x). \quad (14)$$

where g is the subgradient.

The dual cut presented for Bender's decomposition follows the above logic. Treat P_1 as the variable, for the subproblem, if we combine the constraint with the objective function, we have a Lagrangian function:

$$L(P_1, P_2; \lambda) = 2P_2^2 + \lambda(11 - P_1 - P_2) \quad (15)$$

Therefore, $-\lambda(k)$ is a subgradient of $\frac{\partial L}{\partial P_1}$ at k -th step. The deep cut is

$$\begin{aligned} \eta &\geq 2P_2^2(k) - \lambda(k)(P_1 - P_1(k)) \\ &\geq 2P_2^2(k) + \lambda(k)(11 - P_2(k) - P_1) \end{aligned} \quad (16)$$

Consider that there are multiple steps and each step will introduce a cut into the master problem, then the master problem after k step becomes:

$$\min_{P_1} 2P_1^2 + \eta \quad (17)$$

$$\text{s.t. } \eta \geq 4P_2^2(l) + \lambda(l)(11 - P_1 - P_2(l)), \quad l = 1, \dots, k \quad (18)$$

$$6 \leq P_1 \leq 10 \quad (19)$$

The decision variable from the above problem is $P_1(k+1)$. The objective found from the master problem is a lower bound since the dual problem is in the picture. Based on $P_1(k+1)$, we can again solve the subproblem and find $\lambda(k+1)$ and $P_2(k+1)$. Thus the $(k+1)$ -th cut will be added to the master problem.

The procedure will stop when the decision variable values converge. We can also check lower bound and upper bound to determine if we need to stop the procedure. Below is the Matlab code for 20 iteration of Bender's decomposition. The initial value of P_1 is selected as 10 MW. The code provides the master problem with cuts constructed from each iteration.

Fig. 2(a) and Fig. 2(b) present the power output from Generator 1 and the upper and lower bounds.

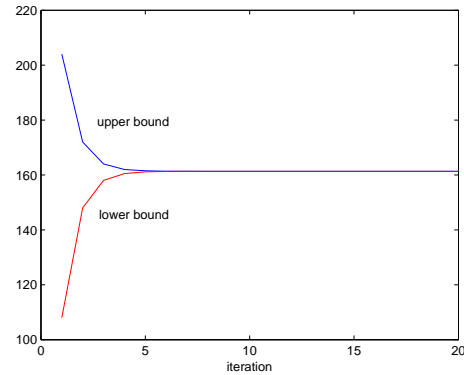
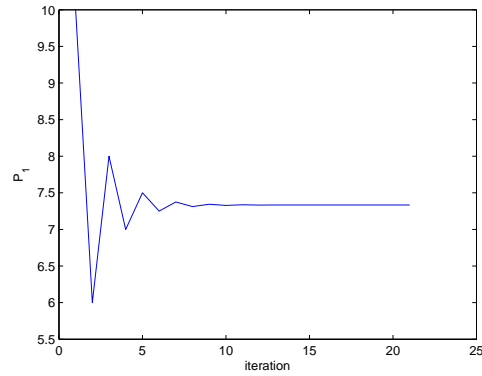


Fig. 2. a) P_1 ; b) upper and lower bounds

III. EXAMPLE 2: MIP SOLVED BY BENDER'S DECOMPOSITION

In the second example, we deal with an MIP problem. Two generators are to serve three switchable loads. The sizes of the loads are 4 MW, 5 MW, 6 MW. The first generator has a cost function of $2P_1$ while the second generator has a cost function of $3P_1$. Suppose that the penalty for switching off a load is

```

x(1)=10;
for i=1:20
cvx_begin
    cvx_solver gurobi
    variable P
    variable g
    minimize(2*P^2 + g)
    subject to
        for j=1:i
            g>=8*(11-x(j))*(x(j)-P)+4*(11-x(j))^2;
        end
        0<=P<=10;
    cvx_end
    x(i+1)= P;
    z_up(i) = 2*x(i)^2 + 4*(11-x(i))^2;
    Z_low(i) = 2*P^2+g;
end

figure(1);
plot(x); ylabel('P_1'); xlabel('iteration');
figure(2);
plot(z_up); hold on; plot(Z_low,'r');xlabel('iteration')

```

Fig. 1. CVX codes for Bender's decomposition.

\$5/MWh. Then the MIP problem is presented as follows.

$$\min_{P_i, u_j} 2P_1 + 3P_2 - 5 \sum_{j=1}^3 u_j D_j \quad (20)$$

$$\text{s.t. } u_i \in \{0, 1\} \quad (21)$$

$$0 \leq P_1 \leq 5 \quad (22)$$

$$0 \leq P_2 \leq 8 \quad (23)$$

$$P_1 + P_2 = \sum_{j=1}^3 u_j D_j \quad (24)$$

The solution of the problem can be examined and found as $P_1 = 5$, $P_2 = 6$ and D_2 and D_3 are switched on.

The integer variables u_j and the continuous variables P_i will be separated. When u_j is fixed, P_i will be found. The objective is an upper bound. Based on the subproblem's Lagrangian dual, we will generate a cut:

$$\eta \geq \lambda(k) \left(\sum_{j=1}^3 u_j D_j - P_1(k) - P_2(k) \right) + 2P_1(k) + 3P_2(k) \quad (25)$$

The master problem will determine u_j based on:

$$\min_{u_j} -5 \sum_{j=1}^3 u_j D_j + \eta \quad (26)$$

$$\text{s.t. } u_j \in \{0, 1\} \quad (27)$$

$$0 \leq \sum_{j=1}^3 u_j D_j \leq 13 \quad (28)$$

$$\eta \geq \lambda(l) \left(\sum_{j=1}^3 u_j D_j - P_1(l) - P_2(l) \right) + 2P_1(l) + 3P_2(l), \quad (29)$$

$$l = 1, \dots, k.$$

Start with $u = [0, 0, 0]^T$, we have $P_1 = P_2 = 0$ and $\lambda = 2$. The cut generated is

$$\eta \geq 2 \left(\sum_{j=1}^3 u_j D_j \right) \quad (30)$$

The main problem becomes:

$$\min_{u_j} -5 \sum_{j=1}^3 u_j D_j + \eta \quad (31)$$

$$\text{s.t. } u_j \in \{0, 1\} \quad (32)$$

$$0 \leq \sum_{j=1}^3 u_j D_j \leq 13 \quad (33)$$

$$\eta \geq 2 \left(\sum_{j=1}^3 u_j D_j \right) \quad (34)$$

The solution is $u = [0, 1, 1]^T$. Based on this solution, we find $P_1 = 5$ and $P_2 = 6$, $\lambda = 3$. The master problem becomes:

$$\min_{u_j} -5 \sum_{j=1}^3 u_j D_j + \eta \quad (35)$$

$$\text{s.t. } u_i \in \{0, 1\} \quad (36)$$

$$0 \leq \sum_{j=1}^3 u_j D_j \leq 13 \quad (37)$$

$$\eta \geq 2 \left(\sum_{j=1}^3 u_j D_j \right) \quad (38)$$

$$\eta \geq 3 \left(\sum_{j=1}^3 u_j D_j - 11 \right) + 28 \quad (39)$$

The solution is again $u = [0, 1, 1]^T$. Convergence reaches.

IV. EXAMPLE 3: MIP SOLVED BY B&B

In this section, the above MIP problem will be solved by B&B. B&B starts with a feasible solution to find an upper bound. Then it employs LP relaxation to find the lower bounds. Next, B&B will separate the feasible region into subregions and for each subregion, LP relaxation will be applied. That way, a lower bound will be found for each subregion. The subregions will be pruned based on optimality and bound.

In this MIP example, we first find a feasible solution $u = [0, 0]^T$ and $P = [0, 0, 0]^T$. For the original problem, an upper bound is found as $z_u = 0$. Next, LP relaxation is applied for the original problem:

$$\min_{P_i, u_j} 2P_1 + 3P_2 - 5 \sum_{j=1}^3 u_j D_j \quad (40)$$

$$\text{s.t. } 0 \leq u_j \leq 1, j = 1, \dots, 3 \quad (41)$$

$$0 \leq P_1 \leq 5 \quad (42)$$

$$0 \leq P_2 \leq 8 \quad (43)$$

$$P_1 + P_2 = \sum_{j=1}^3 u_j D_j \quad (44)$$

The value of the relaxed problem is the lower bound of the original problem: $z_{lower} = -31$.

The feasible region will be separated based on u_1 . For subregion 1, $u_1 = 0$ (Node 1) and for subregion 2, $u_1 = 1$ (Node 2). For Node 1, we found a feasible solution $z_u = -27$ when $u = [0, 1, 1]^T$. Apply LP relaxation, the lower bound

$z_{lower} = -27$ can be computed from the following problem.

$$\min_{P_i, u_j} 2P_1 + 3P_2 - 5 \sum_{j=1}^3 u_j D_j \quad (45)$$

$$\text{s.t. } 0 \leq u_j \leq 1, j = 1, \dots, 3 \quad (46)$$

$$0 \leq P_1 \leq 5 \quad (47)$$

$$0 \leq P_2 \leq 8 \quad (48)$$

$$P_1 + P_2 = \sum_{j=1}^3 u_j D_j \quad (49)$$

$$u_1 = 0 \quad (50)$$

The solution of the above problem is $u = [0, 1, 1]^T$. This is a feasible solution. Therefore, the upper bound is also -27. Node 1 has same lower bound and upper bounds. Therefore Node 1 is pruned for optimality.

We then move forward to Node 2 to examine the rest of the feasible regions. For Node 2, apply LP relaxation, the lower bound is found as -31. The upper bound for $u = [1, 0, 1]^T$ is -25. Separate Node 2 into two subregions ($u_1 = 1, u_2 = 0$, Node 3) and ($u_1 = 1, u_2 = 1$, Node 4). The lower bounds for Node 3 and Node 4 are -25 and -31. The solution for the LP relaxed problem of Node 3 is a feasible solution. Therefore, the upper bound of Node 3 is -25. Node 3 is pruned for optimality. Node 4 is separated into Node 5 and Node 6. We found that Node 6 has infeasible region and Node 5 has both lower and upper bounds as -23.

Comparing the three nodes 1, 3, and 5, the optimal solution is Node 1 where $u = [0, 1, 1]^T$ and $P = [5, 6]^T$.

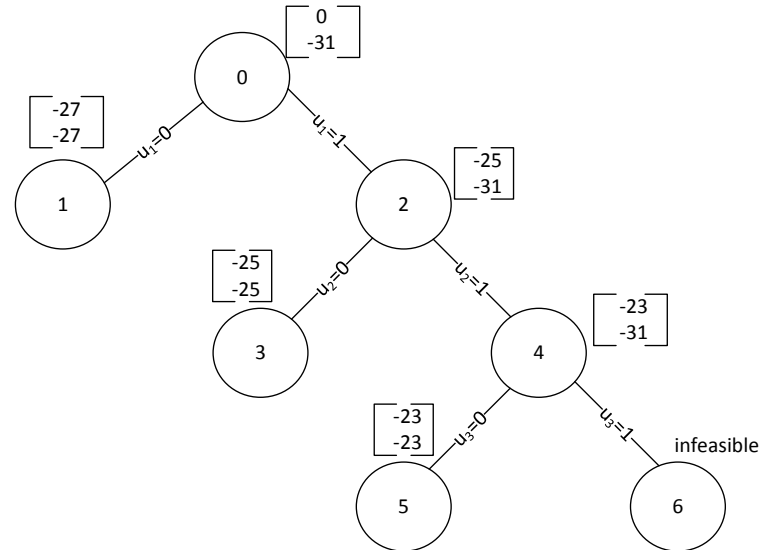


Fig. 3. Branch and bound.

V. CONCLUSION

This tutorial gives three simple examples to help readers to understand Bender's decomposition and Branch and Bound.

REFERENCES

- [1] M. Grant, S. Boyd, and Y. Ye, "Cvx: Matlab software for disciplined convex programming," 2008.
- [2] L. Kuznia, B. Zeng, G. Centeno, and Z. Miao, "Stochastic optimization for power system configuration with renewable energy in remote areas," *Annals of Operations Research*, vol. 210, no. 1, pp. 411–432, 2013.
- [3] M. Shahidehpour and Y. Fu, "Tutorial: benders decomposition in restructure power systems," 2005.
- [4] S. Boyd and L. Vandenberghe, "Localization and cutting-plane methods," *Lecture notes, Stanford University*, www.stanford.edu/class/ee392o/localization-methods.pdf, 2007.